Emma Lehmer proved in 1938 in a paper which later became a classic a congruence mod p (a prime > 3) between a sum of reciprocals of squares of integers, denoted $S_4(p)$, and the Euler number E_{p-3}. As a consequence of this result, $S_4(p)$ vanishes mod p exactly when $E_{p-3} \equiv 0 \pmod{p}$, that is, $(p, p-3)$ is an E-irregular pair. This property of the prime p, originally interesting in view of the Fermat problem, appeared to be quite rare: up to 2.9 million there are only two such primes, 149 and 241. (Note by the way that among the ordinary irregular primes, the corresponding property also seems to be quite exceptional, the first prime with $B_{p-3} \equiv 0 \pmod{p}$ being 16843.)

The authors generalize the Lehmer congruence to arbitrary moduli n. Denote the sum in question by $S_4(n)$. Now the Euler number appearing in the congruence is $E_{\phi(n)-2}$. In the meantime, the Lehmer congruence had been already generalized to prime power moduli (*T.X. Cai* et al., Acta Arith. 130, 2007, 203–214). The present authors' work which requires quite involved computations and manipulations shows great skill and expertise. In my opinion, the main importance of their result lies in the fact, proved as a corollary, that the vanishing mod p of $S_4(n)$ essentially implies, for odd n, that n is divisible by a prime p satisfying $E_{p-3} \equiv 0 \pmod{p}$ (with some restrictions and modifications). This gives a new interesting insight on these special primes p.

There are also some numerical examples and tables illustrating the corollary, as well as some considerations concerning properties of $S_4(n)$ for even moduli n.

The paper is clearly and carefully written, except that the very last pages contain a few misprints or lapses. All the errors are easy to correct, however; see the detailed comments below.

The paper is doubtless worth publishing. Its publication in Acta Arithmetica is supported by the fact that Cai et al. in their paper mentioned above and published in AA say that they were not able to do the generalization the present authors now did.

Detailed Comments

Page 3, last sentence in the proof of Lemma 1: 'power of a' should probably be 'exponent of a'. Secondly, is there some need for A? Why not take $A = 1$?

Page 7, line -6: The big parentheses could be dropped.

Page 10, line 8: Add 'cdot' between 3^{-2} and 10.

Page 14, first line of Section 6: (3.4) should be (3.3).

Page 16, last line of Section 6: (6.4) should be (6.5).

Page 16, Eq. (7.3): This congruence is said to be quoted from an article by Z.-H. Sun mentioned in the bibliography. I had no access to that article, but something seems to be wrong. If the summation is really extended to $(p-1)/4$, then one should assume that $p \equiv 1 \pmod{4}$.

Page 16, the line after (7.4): $q_a(n)$ should read $q_n(a)$.